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A Generalized Liapunov Inequality

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1. INTRODUCTION

For the study of the qualitative nature of solutions of ordinary linear differential equations of the second order, a very useful inequality is

$$\int_a^b [\eta'(t)]^2 \, dt \geqslant (4/[b-a]) \, \eta^2(s), \qquad s \in (a, b), \tag{1.1}$$

for arbitrary real-valued absolutely continuous functions η on $[a, b]$, with η' of integrable square and $\eta(a) = 0 = \eta(b)$; moreover, if $\eta \not\equiv 0$ on $[a, b]$ the equality holds in (1.1) only if $s = (a+b)/2$ and

$$\eta(t) = \eta(s)\{1 - |(2t - a - b)/(b - a)|\}.$$

In particular, with the aid of this inequality one may show that if q is a real-valued function such that the differential equation,

$$u''(t) + q(t)\, u(t) = 0, \tag{1.2}$$

has a nonidentically vanishing real-valued solution possessing two distinct zeros on $[a, b]$ then $q^+ = \frac{1}{2}[q + |q|]$ must satisfy the Liapunov inequality

$$\int_a^b q^+(t) \, dt > (4/[b-a]). \tag{1.3}$$

In previous papers [2, Theorems 2.3, 4.2 and 4, Theorems 3.1, 3.2], the author has given certain matrix generalizations of (1.1) and (1.3). The purpose of the present paper is to present still further matrix generalizations of these inequalities. In particular, through the use of generalized linear differential systems of the sort considered by the author [3; 6], in Section 2 there are presented results which provide basic interpretations of the involved constants in various inequalities of Liapunov type.

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Section 3 is devoted to Liapunov type inequalities for a real self-adjoint scalar differential equation of order $2n$, written as

$$L[y](t) \equiv \sum_{j=0}^{n} (-1)^j \{p_j(t)\, y^{[j]}(t)\}^{[j]} = 0, \tag{1.4}$$

where for $j = 0, 1,..., n$ the real-valued function p_j has continuous derivatives of the first j orders on a compact interval $[a, b]$, with $p_n(t) > 0$ on this interval, and (1.4) disconjugate on $[a, b]$ in the sense that there exists no nonidentically vanishing solution y of this equation for which there are distinct values t_1, t_2 on $[a, b]$ with $y^{[\alpha-1]}(t_1) = 0 = y^{[\alpha-1]}(t_2)$, $(\alpha = 1,..., n)$. As a direct generalization of the classical Liapunov inequality, it is shown that if q is a real-valued function such that the equation,

$$L[y](t) - q(t)\, y(t) = 0, \tag{1.5}$$

has a nonidentically vanishing solution, which possesses on $[a, b]$ two distinct zeros of order at least n, then

$$\int_a^b q^+(t)\, dt > 1/\mathrm{Max}\{g(s, s) \mid s \in [a, b]\}, \tag{1.6}$$

where $g(t, s)$, $(t, s) \in [a, b] \times [a, b]$, is the Green's function for the incompatible boundary problem

$$L[y](t) = 0, \qquad y^{[\alpha-1]}(a) = 0 = y^{[\alpha-1]}(b), \qquad (\alpha = 1,..., n). \tag{1.7}$$

Finally, Section 4 is concerned with the special equation $(-1)^n y^{[2n]} = 0$.

Matrix notation is used throughout; in particular, matrices of one column are called vectors, and for a vector (w_α), $(\alpha = 1,..., n)$, the norm $|w|$ is given by $(|w_1|^2 + \cdots + |w_n|^2)^{1/2}$; the linear vector space of ordered n-tuples of complex numbers, with complex scalars, is denoted by $\mathbf{C}_n$. The $n \times n$ identity matrix is denoted by E_n, or by merely E when there is no ambiguity, and 0 is used indiscriminately for the zero matrix of any dimensions; the conjugate transpose of a matrix M is denoted by M^*. If M is an $n \times n$ matrix the symbol $|M|$ is used for the supremum of $|Mw|$ on the unit ball $\{w \mid |w| \leqslant 1\}$ of $\mathbf{C}_n$. The notation $M \geqslant N$, $\{M > N\}$, is used to signify that M and N are hermitian matrices of the same dimensions, and $M - N$ is a nonnegative (positive) definite hermitian matrix. If an hermitian matrix function $M(t)$, $t \in [a, b]$, is such that $M(s) - M(t) \geqslant 0$, $(\leqslant 0)$, for

$$a \leqslant s < t \leqslant b,$$

then $M(t)$ is called nonincreasing (nondecreasing) hermitian on $[a, b]$. A matrix function is called continuous, integrable, etc., when each element of the matrix function possesses the specified property.

If a matrix function $M(t)$ is a.c. (absolutely continuous) on $[a, b]$, then $M'(t)$ signifies the matrix of derivatives at values where these derivatives exist, and zero elsewhere. Similarly, if $M(t)$ is (Lebesgue) integrable on $[a, b]$ then $\int_a^b M(t)\, dt$ denotes the matrix of integrals of respective elements of $M(t)$. For a given interval $[a, b]$ the symbols $\mathfrak{C}_{mn}[a, b]$, $\mathfrak{C}^k_{mn}[a, b]$, $\mathfrak{L}_{mn}[a, b]$, $\mathfrak{L}^p_{mn}[a, b]$, $1 < p < \infty$, $\mathfrak{L}^\infty_{mn}[a, b]$, $\mathfrak{A}_{mn}[a, b]$, $\mathfrak{A}^k_{mn}[a, b]$, and $\mathfrak{BV}_{mn}[a, b]$ are used to denote the classes of $m \times n$ matrix functions $M(t) = [M_{\alpha\beta}(t)]$, $(\alpha = 1,..., m; \beta = 1,..., n)$, which are, respectively, continuous, continuous and possessing continuous derivatives of the first k orders, (Lebesgue) integrable, (Lebesgue) measurable and $| M_{\alpha\beta}(t)|^p$ integrable, measurable and essentially bounded, a.c., of class $\mathfrak{C}^{k-1}_{mn}[a, b]$ with $M^{[k-1]}(t) \in \mathfrak{A}_{mn}[a, b]$, and of b.v. (bounded variation) on $[a, b]$. For brevity, the double subscript mn is reduced to merely m for the m-dimensional vector case specified by m, $n = 1$, and both subscripts are omitted in the scalar case $m = 1$, $n = 1$. For $n \geqslant 1$, the subclass of vector functions $y \in \mathfrak{A}_n{}^k[a, b]$ for which $y^{[k]} \in \mathfrak{L}_n{}^2[a, b]$ is denoted by $\mathfrak{A}^{k,2}_n[a, b]$. Also for $n \geqslant 1$ the subclasses of vector functions y belonging to $\mathfrak{C}_n{}^k[a, b]$, $\mathfrak{A}_n{}^k[a, b]$, $\mathfrak{A}^{k,2}_n[a, b]$ for which $y^{[\alpha-1]}(a) = 0 = y^{[\alpha-1]}(b)$, $(\alpha = 1,..., n)$ are denoted by $\mathfrak{C}^k_{n,0}[a, b]$, $\mathfrak{A}^k_{n,0}[a, b]$, $\mathfrak{A}^{k,2}_{n,0}[a, b]$, respectively. If the matrix functions $M(t)$ and $N(t)$ are equal a.e. (almost everywhere) on their interval of definition we write simply $M(t) = N(t)$. If $M(t) \in \mathfrak{BV}_{mn}[a, b]$, $S(t) \in \mathfrak{C}_{rm}[a, b]$, and $T(t) \in \mathfrak{C}_{ns}[a, b]$, then $\int_a^b S(t)[dM(t)]\, T(t)$ denotes the $r \times s$ matrix with elements given by the Riemann-Stieltjes integrals

$$\sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} \int_a^b S_{i\alpha}(t)\, T_{\beta j}(t)\, dM_{\alpha\beta}(t);$$

also, $\int_a^b [dM(t)]\, T(t)$ and $\int_a^b S(t)[dM(t)]$ designate $\int_a^b E_m[dM(t)]\, T(t)$ and $\int_a^b S(t)[dM(t)]\, E_n$, respectively.

2. Basic Inequalities

As in Reid [3, 6, see also 7, Section II.8], we shall consider a self-adjoint generalized differential system

$$\begin{aligned} \Delta[u, v](t) &\equiv -dv(t) + [C(t)\, u(t) - A^*(t)\, v(t)]\, dt + [dM(t)]\, u(t) = 0, \\ L_2[u, v](t) &\equiv u'(t) - A(t)\, u(t) - B(t)\, v(t) = 0, \end{aligned} \tag{2.1}$$

where A, B, C, M are $n \times n$ matrix functions defined on a nondegenerate interval I on the real line which satisfy the following hypothesis.

(1) $B(t) = B^*(t)$, $C(t) = C^*(t)$, $M(t) = M^*(t)$ *for* $t \in I$;

($\mathfrak{H}$) (2) *for arbitrary compact subintervals* $[a, b]$ *of* I *the matrix functions* A, B, C *belong to* $\mathfrak{L}^\infty_{nn}[a, b]$, *and* $M \in \mathfrak{BV}_{nn}[a, b]$.

A solution of (2.1) is understood to be a pair of n-dimensional vector functions u, v which belong to $\mathfrak{A}_n[a, b]$ and $\mathfrak{BB}_n[a, b]$, respectively, with $L_2[u, v] = 0$ on I and the Riemann–Stieltjes integral equation,

$$v(t) = v(\tau) + \int_\tau^t [C(s)\, u(s) - A^*(s)\, v(s)]\, ds + \int_\tau^t [dM(s)]\, u(s), \quad (2.2)$$

holds for $(\tau, t) \in I \times I$.

For $[a, b]$ a compact subinterval of I the symbol $\mathfrak{D}[a, b]$ will denote the class of vector functions $\eta \in \mathfrak{A}_n[a, b]$ for which there is a corresponding $\zeta \in \mathfrak{L}_n{}^2[a, b]$ such that $L_2[\eta, \zeta] = 0$ on $[a, b]$, and the fact that ζ is such an associated vector function will be indicated by $\eta \in \mathfrak{D}[a, b] : \zeta$. The subclass of $\mathfrak{D}[a, b]$ on which $\eta(a) = 0 = \eta(b)$ will be denoted by $\mathfrak{D}_0[a, b]$, with similar meaning for $\eta \in \mathfrak{D}_0[a, b] : \zeta$. If $(\eta_\alpha, \zeta_\alpha) \in \mathfrak{L}_n{}^2[a, b] \times \mathfrak{L}_n{}^2[a, b]$ for $\alpha = 1, 2$, we denote by $J[\eta_1 : \zeta_1, \eta_2 : \zeta_2\, ; a, b]$ the expression

$$\int_a^b \{\zeta_2{}^*(t)\, B(t)\, \zeta_1(t) + \eta_2{}^*(t)\, C(t)\, \eta_1(t)\}\, dt + \int_a^b \eta_2{}^*(t)[dM(t)]\, \eta_1(t), \quad (2.3_0)$$

which clearly defines an hermitian functional on $\mathfrak{L}_n{}^2[a, b] \times \mathfrak{L}_n{}^2[a, b]$. If $\eta_\alpha \in \mathfrak{D}[a, b] : \zeta_\alpha$, $(\alpha = 1, 2)$, the vector functions ζ_α are in general not determined uniquely. The vector functions $B\zeta_\alpha$ are uniquely determined elements of $\mathfrak{L}_n{}^2[a, b]$, however, and (2.3_0) defines a functional of (η_1, η_2) on

$$\mathfrak{D}[a, b] \times \mathfrak{D}[a, b].$$

Consequently, in this case the symbol for (2.3_0) is reduced to $J[\eta_1, \eta_2\, ; a, b]$. As usual, for $\eta \in \mathfrak{D}[a, b] : \zeta$ we write $J[\eta; a, b]$ in place of the more complicated notation $J[\eta, \eta; a, b]$ for the Dirichlet functional,

$$J[\eta; a, b] = \int_a^b \{\zeta^*(t)\, B(t)\, \zeta(t) + \eta^*(t)\, C(t)\, \eta(t)\}\, dt + \int_a^b \eta^*(t)[dM(t)]\, \eta(t). \quad (2.3)$$

As in [3, Theorem 2.2], we have that if $[a, b] \subset I$ and $u \in \mathfrak{A}_n[a, b]$, then there exists a v such that (u, v) is a solution of (2.1) on $[a, b]$ iff there exists a $v_1 \in \mathfrak{L}_n{}^2[a, b]$ such that $u \in \mathfrak{D}[a, b] : v_1$ and $J[u : v_1, \eta : \zeta; a, b] = 0$, for $\eta \in \mathfrak{D}_0[a, b] : \zeta$. Finally, $\mathfrak{H}_+[a, b]$ will denote the condition that $J[\eta; a, b]$ is positive definite on $\mathfrak{D}_0[a, b]$; that is, if $\eta \in \mathfrak{D}_0[a, b] : \zeta$, then $J[\eta; a, b] \geqslant 0$, with equality holding only if $\eta \equiv 0$ on $[a, b]$, in which case $B\zeta = 0$ on this interval.

Two distinct values t_1 and t_2 on I are said to be *conjugate* with respect to (2.1) if there exists a solution (u, v) of this system with $u \not\equiv 0$ on the subinterval with endpoints t_1 and t_2, while $u(t_1) = 0 = u(t_2)$. The system (2.1) is called *disconjugate* on a subinterval I_0 of I provided no two distinct points of I_0 are conjugate. As in [3, Theorem 5.1], under hypothesis ($\mathfrak{H}$) the functional $J[\eta; a, b]$ satisfies $\mathfrak{H}_+[a, b]$ iff $B(t) \geqslant 0$ for t a.e. on $[a, b]$, and (2.1) is disconjugate on $[a, b]$. Also, by [3, Theorem 5.2], if ($\mathfrak{H}$) and $\mathfrak{H}_+[a, b]$ hold and $[c, d]$ is a nondegenerate subinterval of $[a, b]$, then for $\eta \in \mathfrak{D}[c, d]$ there exists a solution (u, v) of (2.1) such that $u(c) = \eta(c)$, $u(d) = \eta(d)$; moreover, $J[\eta; c, d] \geqslant J[u; c, d]$ with equality iff $\eta = u$ and $B[\zeta - v] = 0$ on $[c, d]$. Furthermore, in view of the results of Theorems 2.2 and 5.3 of [3] we have the following property of systems (2.1).

LEMMA 2.1. *Suppose that hypothesis ($\mathfrak{H}$) holds, and $[a, b]$ is a nondegenerate subinterval of I with $J[\eta; a, b]$ nonnegative on $\mathfrak{D}_0[a, b]$. Then either*

(i) *there exists a solution (u, v) of (2.1) with $u \not\equiv 0$ on $[a, b]$ and $u(a) = 0 = u(b)$, in which case $u \in \mathfrak{D}_0[a, b] : v$ and $J[u; a, b] = 0$, or*

(ii) *there exists a $\kappa > 0$ such that if Π is an $n \times n$ nondecreasing hermitian matrix function which is not constant on $[a, b]$, then*

$$J[\eta; a, b] \geqslant (\kappa / V[a, b; \Pi]) \int_a^b \eta^*(t)[d\Pi(t)]\, \eta(t) \qquad \text{for} \quad \eta \in \mathfrak{D}_0[a, b], \quad (2.4)$$

where $V[a, b; \Pi]$ is the supremum of $\sum_{\alpha=1}^{m} | \Pi(t_\alpha) - \Pi(t_{\alpha-1})|$ for all subdivisions $a = t_0 < t_1 < \cdots < t_m = b$ of $[a, b]$.

Moreover, if $\eta \in \mathfrak{D}_0[a, b]$ is such that $J[\eta; a, b] = 0$ then there exists a $\zeta \in \mathfrak{BV}_n[a, b]$ such that $(u, v) = (\eta, \zeta)$ is a solution of (2.1) on $[a, b]$.

Finally, a basic result for the present discussion is that of the following theorem, corresponding to the result of the Corollary to Theorem 5.3 of [3].

THEOREM 2.1. *Suppose that hypothesis ($\mathfrak{H}$) holds, and Π is a nondecreasing hermitian matrix function on a nondegenerate subinterval $[a, b]$ of I such that*

$$\mathfrak{D}_0[a, b; \Pi] = \left\{\eta \mid \eta \in \mathfrak{D}_0[a, b], \int_a^b \eta^*(t)[d\Pi(t)]\, \eta(t) = 1\right\} \qquad (2.5)$$

is nonempty. Whenever hypothesis $\mathfrak{H}_+[a, b]$ is satisfied, and

$$\lambda_1 = \inf\{J[\eta; a, b] \mid \eta \in \mathfrak{D}_0[a, b; \Pi]\}, \qquad (2.6)$$

then for $\lambda = \lambda_1$ there exists a solution $(u, v) = (u_1, v_1)$ of the boundary problem

$$\begin{aligned} \Delta[u, v](t) - \lambda[d\Pi(t)]\, u(t) &= 0, \\ L_2[u, v](t) &= 0, \\ u(a) = 0 &= u(b), \end{aligned} \qquad (2.7)$$

with

$$\int_a^b u_1^*(t)[d\Pi(t)]\, u_1(t) = 1, \qquad J[u_1\,; a, b] = \lambda_1\,. \tag{2.8}$$

Moreover, $\lambda = \lambda_1$ *is the largest constant such that*

$$J[\eta; a, b] \geqslant \lambda \int_a^b \eta^*(t)[d\Pi(t)]\, \eta(t), \qquad \textit{for} \quad \eta \in \mathfrak{D}_0[a, b]. \tag{2.9}$$

In view of hypothesis $\mathfrak{H}_+[a, b]$, the value λ_1 defined by (2.6) is clearly nonnegative. As in Section 5 of [3], application of the result of the above Lemma 5.1 to the functional $J[\eta; a, b] - \lambda_1 \int_a^b \eta^*(t)[d\Pi(t)]\, \eta(t)$ yields the existence of a solution $(u_1\,, v_1)$ of (2.7) for $\lambda = \lambda_1$, with $u_1 \not\equiv 0$ on $[a, b]$. Using the usual integration by parts procedure, it follows that if (u, v) is solution of (2.7) for a value λ then

$$J[u; a, b] = \lambda \int_a^b u^*(t)[d\Pi(t)]\, u(t). \tag{2.10}$$

Since u_1 is a nonidentically vanishing vector function in $\mathfrak{D}_0[a, b]$ we have $J[u_1\,; a, b] > 0$, and hence $\lambda_1 > 0$ and $\int_a^b u_1^*(t)[d\Pi(t)]\, u_1(t) > 0$. Consequently, the solution $(u_1\,, v_1)$ of (2.7) for $\lambda = \lambda_1$ may be normed to satisfy $\int_a^b u_1^*(t)[d\Pi(t)]\, u_1(t) = 1$, in which case the second relation of (2.8) is a direct consequence of (2.10). The fact that $\lambda = \lambda_1$ is the largest constant for which (2.9) is valid is a direct consequence of the condition $\mathfrak{H}_+[a, b]$ and the definition (2.6) of λ_1.

The system (2.1) is said to be *normal* on a subinterval I_0 of I if $(u(t) \equiv 0, v(t))$ is a solution of (2.1) on I_0 only if also $v(t) \equiv 0$ on I_0, so that (u, v) is the identically vanishing solution of (2.1). Now if Π is a nondecreasing hermitian matrix function on $[a, b]$, and u is a continuous n-dimensional vector function on $[a, b]$, then $\int_a^b u^*[d\Pi]u \geqslant 0$. Moreover, if $\int_a^b u^*[d\Pi]u = 0$, it then follows that $\int_a^b \eta^*[d\Pi]u = 0$ for arbitrary continuous n-dimensional vector functions η, from which it follows that $\int_a^s [d\Pi(t)]\, u(t) = 0$, for $s \in [a, b]$. Consequently, if (2.1) is a normal system satisfying hypotheses $(\mathfrak{H})$ and $\mathfrak{H}_+[a, b]$, it follows that all proper values of (2.7) are positive, and if (u, v) is a proper solution of this boundary problem corresponding to the proper value λ, then $\int_a^b u^*[d\Pi]u > 0$. In this case the value λ_1 defined by (2.6) may be described also as the smallest proper value of the boundary problem (2.7). If $(u_1\,, v_1)$ and $(u_2\,, v_2)$ are solutions of (2.7) for corresponding values λ_1 and λ_2, it follows that $\int_a^b u_1^*[d\Pi]\, u_2 = 0$. Consequently if Π is a monotone nondecreasing hermitian function on $[a, b]$ for which there is a positive integer k such that k is the largest integer for which there exists a linear subspace of $\mathfrak{C}_n[a, b]$ on which the functional $\int_a^b u^*[d\Pi]u$ is positive definite, then in case hypotheses $(\mathfrak{H})$ and $\mathfrak{H}_+[a, b]$ are satisfied there are at most k proper values of (2.7), with each proper value being counted a number of times equal to its multiplicity. In the special instance wherein $k = 1$ we have that (2.7) possesses a single proper value, and the λ_1 of Theorem 2.1 may be characterized as *the unique* proper value of (2.7).

Now suppose that hypotheses $(\mathfrak{H})$ and $\mathfrak{H}_+[a, b]$ hold, and K is an $n \times n$ nonnegative hermitian matrix such that for $s \in (a, b)$ there exists a vector function $\eta \in \mathfrak{D}_0[a, b]$ such that $\eta^*(s)\, K\eta(s) > 0$. In particular, this condition holds for an arbitrary $n \times n$ nonnegative hermitian matrix K which is not identically zero whenever (2.1) satisfies the further condition of being normal on all subintervals $[a, s]$ and $[s, b]$ with $s \in (a, b)$. Now for $s \in (a, b)$, let Π_s be a nondecreasing hermitian matrix function on $[a, b]$ such that

$$\Pi_s(t) = 0, \quad t \in [a, s); \qquad \Pi_s(t) = K, \quad t \in (s, b]. \tag{2.11}$$

From Theorem 2.1 it follows that for $s \in (a, b)$ the constant,

$$\mu_1(s) = \inf\{J[\eta; a, b] \mid \eta \in \mathfrak{D}_0[a, b; \Pi_s],\tag{2.12}$$

is the largest value such that

$$J[\eta; a, b] \geqslant \mu_1(s)\, \eta^*(s)\, K\eta(s), \qquad \text{for} \quad \eta \in D_0[a, b], \tag{2.13}$$

and consequently, if

$$\mu_1 = \inf\{\mu_1(s) \mid s \in (a, b)\}, \tag{2.14}$$

then $\mu = \mu_1$ is the largest value such that

$$J[\eta; a, b] \geqslant \mu_1\, \eta^*(s)\, K\eta(s), \qquad \text{for} \quad s \in (a, b) \quad \text{and} \quad \eta \in \mathfrak{D}_0[a, b]. \tag{2.15}$$

For $M(t) \equiv 0$ the system (2.1) becomes the ordinary differential equation system

$$\begin{aligned} L_1[u, v](t) &\equiv -v'(t) + C(t)\, u(t) - A^*(t)\, v(t) = 0, \\ L_2[u, v](t) &\equiv u'(t) - A(t)\, u(t) - B(t)\, v(t) = 0, \end{aligned} \tag{2.16}$$

and the results of Theorem 2.1 and the above remarks yield the following results for the associated functional

$$J_0[\eta; a, b] = \int_a^b [\zeta^*(t)\, B(t)\, \zeta(t) + \eta(t)\, C(t)\, \eta(t)]\, dt. \tag{2.17}$$

THEOREM 2.2. *Suppose that the matrix function M is identically zero, the matrix functions A, B, C satisfy $(\mathfrak{H})$ and $B(t) \geqslant 0$ for t a.e. on I, while for a given nondegenerate compact subinterval $[a, b]$ of I the system (2.16) is disconjugate on $[a, b]$, and normal on all subintervals $[a, s]$, $[s, b]$ with $s \in (a, b)$.*

If K is a nonzero $n \times n$ matrix that is nonnegative hermitian, and for $s \in (a, b)$ the symbol Π_s denotes a nondecreasing hermitian matrix function on $[a, b]$ satisfying (2.11), *then for $s \in (a, b)$ the largest constant $\mu_1 = \mu_1(s)$ such that*

$$J_0[\eta; a, b] \geqslant \mu_1(s)\, \eta^*(s)\, K\eta(s) \qquad \text{for} \quad \eta \in \mathfrak{D}_0[a, b] \tag{2.18}$$

is the smallest value μ_1 for which there exists a nonidentically vanishing pair of vector functions u_1, v_1 satisfying the following conditions:

(a) (u_1, v_1) *is a solution of* (2.16) *on each of the subintervals* $[a, s)$ *and* $(s; b]$;

(b) u_1 *is* a.c. *on* $[a, b]$, *and*

$$u_1(a) = 0 = u_1(b); \tag{2.19}$$

(c) v_1 *is of* b.v. *on* $[a, b]$, *and*

$$v_1(s^+) - v_1(s^-) + \mu_1 K u_1(s) = 0. \tag{2.20}$$

Moreover, $\mu_1(s) > 0$ and $Ku_1(s) \neq 0$. In particular, if K is of rank one then $\mu_1(s)$ is the unique real value μ_1 for which there is a nonidentically vanishing pair of vector functions (u_1, v_1) satisfying the conditions (a), (b), (c), *and any pair (u, v) satisfying these conditions is a constant multiple of (u_1, v_1).*

Now whenever the system (2.16) is disconjugate on $[a, b]$ the self-adjoint boundary problem (2.16), (2.19) is incompatible, and hence there exists a corresponding Green's matrix. In particular, (see Reid [5, Section III: 7 and especially Theorem VII: 8.2), there exist $n \times n$ matrix functions $G(t, s)$, $G_0(t, s)$ for $(t, s) \in \square = [a, b] \times [a, b]$ such that if $\phi \in \mathfrak{L}_n^2[a, b]$ then the unique solution (u, v) of the differential system

$$L_1[u, v](t) = \phi(t), \qquad L_2[u, v](t) = 0, \qquad u(a) = 0 = u(b), \tag{2.21}$$

is given by

$$\begin{aligned} u(t) &= \int_a^b G(t, s)\, \phi(s)\, ds, \\ v(t) &= \int_a^b G_0(t, s)\, \phi(s)\, ds. \end{aligned} \tag{2.22}$$

Moreover, the matrix functions G, G_0 are characterized by the following properties.

(i) *G is continuous in (t, s) on $\square$, is* a.c. *in each argument on $[a, b]$ for fixed values of the other argument, and $G(t, s) \equiv G^*(s, t)$ on $\square$.*

(ii) *G_0 is continuous in (t, s) on each of the triangular domains $\Delta_1 = \{(t, s) \mid (t, s) \in \Delta, s < t\}$ and $\Delta_2 = \{(t, s) \mid (t, s) \in \square, t < s\}$, is bounded on $\square$,* *while the restriction of* G_0 *to* Δ_α, $(\alpha = 1, 2)$, *has a finite limit at each* (t, t) *with* $t \in [a, b]$, *and*

$$G_0(s^+, s) - G_0(s^-, s) + E = 0, \qquad \textit{for} \quad s \in (a, b).$$

(iii) *If* $s \in (a, b)$, *and* ξ *is an arbitrary n-dimensional vector, then* $(u(t), v(t)) = (G(t, s)\xi, G_0(t, s)\xi)$ *is the unique pair of n-dimensional vector functions satisfying the following conditions*:

(a) (u, v) *is a solution of* (2.6) *on each of the subintervals* $[a, s)$ *and* $(s, b]$;

(b) u *is* a.c. *on* $[a, b]$, *with* $u(a) = 0 = u(b)$;

(c) v *is of* b.v. *on* $[a, b]$, *with*

$$v(s^+) - v(s^-) + \xi = 0.$$

If (u_α, v_α), $(\alpha = 1, 2)$, are solutions of (2.16), then the function

$$\{u_1, v_1; u_2, v_2\} = v_2{}^*u_1 - u_2{}^*v_1$$

is constant on I; if this constant is zero then (u_1, v_1) and (u_2, v_2) are said to be *conjugate* or *conjoined* solutions of (2.16). Consequently, if $(U, V) = (U_a, V_a)$ and $(U, V) = (U_b, V_b)$ are $n \times n$ matrix functions which are solutions of the matrix differential equations $L_1[U, V] = 0$, $L_2[U, V] = 0$, while $U_a(a) = 0$, $V_a(a)$ is nonsingular and $U_b(b) = 0$, $V_b(b)$ is nonsingular, then $\{U_a, V_a; U_a, V_a\} = 0$, $\{U_b, V_b; U_b, V_b\} = 0$, and

$$M = \{U_a, V_a; U_b, V_b\} = V_b{}^*U_a - U_b{}^*V_a$$

is a constant $n \times n$ matrix function on I_0.

In particular, if (2.16) is disconjugate on $[a, b]$ then M is nonsingular, and upon replacing (U_b, V_b) by $(\hat{U}_b, \hat{V}_b) = (-U_bM^{*-1}, -V_bM^{*-1})$ we have that (U_a, V_a) and $(\hat{U}_b, \hat{V}_b)$ are solutions of the matrix differential system with $U_a(a) = 0$, $V_a(a)$ nonsingular, $\hat{U}_b(b) = 0$, $\hat{V}_b(b)$ nonsingular, and

$$\{U_a, V_a; \hat{U}_b, \hat{V}_b\} = -E.$$

One may verify that the above defined matrix functions G, G_0 have the following representations:

$$\begin{aligned}(G(t, s), G_0(t, s)) &= (U_a(t)\,\hat{U}_b{}^*(s), V_a(t)\,\hat{U}_b{}^*(s)) \\ &= (-U_a(t)\,M^{-1}U_b{}^*(s), -V_a(t)\,M^{-1}U_b{}^*(s)) \\ &\qquad\qquad \text{for} \quad a \leqslant t < s \leqslant b;\end{aligned}$$

$$\begin{aligned}(G(t, s), G_0(t, s)) &= (\hat{U}_b(t)\,U_a{}^*(s), \hat{V}_b(t)\,U_a{}^*(s)) \\ &= (-U_b(t)\,M^{*-1}U_a{}^*(s), -V_b(t)\,M^{*-1}U_a{}^*(s)) \\ &\qquad\qquad \text{for} \quad a \leqslant s < t \leqslant b.\end{aligned}$$

Whenever $B(t) \geqslant 0$ for t a.e. on I, and (2.16) is disconjugate on $[a, b] \subset I$, then for $\phi \in \mathfrak{L}_n{}^2[a, b]$ and (u, v) the solution of (2.21) given by (2.22) we have

$$\begin{aligned} 0 &\leqslant J[u; a, b] \\ &= u^* v \,|_a^b + \int_a^b u^*(t) L_1[u, v](t)\, dt \\ &= \int_a^b \int_a^b \phi^*(t)\, G(t, s)\, \phi(s)\, dt\, ds. \end{aligned}$$

In view of the arbitrariness of ϕ, it then follows readily that $G(s, s) \geqslant 0$ for $s \in [a, b]$, in particular, $G_{rr}(s, s) \geqslant 0$, for $s \in [a, b]$ and $r = 1,..., n$.

Now suppose that the system (2.1) satisfies the hypotheses appearing in the statement of Theorem 2.2, and for $r = 1, 2,..., n$, let K_r denote the $n \times n$ hermitian matrix $[K_{r;\alpha,\beta}]$, $(\alpha, \beta = 1,..., n)$, with $K_{r;\alpha,\beta} = \delta_{\alpha r}\delta_{\beta r}$. Then $K = K_r$ satisfies the conditions specified in Theorem 2.2, and hence the largest constant $\mu_1 = \mu_{1r}(s)$ for which $J_0[\eta; a, b] \geqslant \mu_1 \mid \eta_r(s)|^2$ for $\eta \in \mathfrak{D}_0[a, b]$ is characterized as the unique real value for which there is a nonidentically vanishing pair of vector functions $(u_1, v_1) = (u_1{}^r, v_1{}^r)$ satisfying conditions (a), (b), (c) of Theorem 2.2. In particular, the r-th component u_{r1}^r of $u_1{}^r$ is nonzero, and the vector functions,

$$u(t) = [\mu_{1r}(s)\, u_{r1}^r(s)]^{-1}\, u_1{}^r(t),\ v(t) = [\mu_{1r}(s)\, u_{r1}^r(s)]^{-1}\, v_1{}^r(t),$$

possess the definitive properties listed above for the pair of vector functions $G(t, s)\, e^{(r)}$, $G_0(t, s)\, e^{(r)}$, where $e^{(r)}$ is the unit vector $(\delta_{\alpha r})$. Consequently, $G_{rr}(s, s) = 1/\mu_{1r}(s)$, and as $G_{rr}(a, a) = 0 = G_{rr}(b, b)$, we have the following result.

THEOREM 2.3. *If the system* (2.1) *satisfies the hypothesis appearing in the statement of Theorem* 2.2, *and* G, G_0 *are the matrix functions possessing the resolvent property* (2.22), *then for* $s \in (a, b)$ *and* $r = 1, 2,..., n$ *we have that* $G_{rr}(s, s) > 0$ *and* $\mu_{1r}(s) = 1/G_{rr}(s, s)$ *is the largest value such that*

$$J_0[\eta; a, b] \geqslant \mu_{1r}(s) \mid \eta_r(s)|^2 \qquad \textit{for} \quad \eta \in \mathfrak{D}_0[a, b]. \tag{2.23}$$

Moreover, equality holds in (2.23) *iff* $\eta(t)$ *is of the form* $\kappa G(t, s)\, e^{(r)}$ *for* $t \in [a, b]$. *Correspondingly,*

$$\mu_{1r} = \operatorname{Min}\{\mu_{1r}(s) \mid s \in (a, b)\} = 1/\operatorname{Max}\{G_{rr}(s, s) \mid s \in [a, b]\}$$

is the largest value such that

$$J_0[\eta; a, b] \geqslant \mu_{1r} \mid \eta_r(s)|^2 \qquad \textit{for} \quad \eta \in \mathfrak{D}_0[a, b] \quad \textit{and} \quad s \in (a, b). \tag{2.24}$$

3. A Liapunov Inequality for Higher Order Differential Equations

As a special instance of the results of the preceding section, we shall consider the self-adjoint scalar differential equations of order $2n$, written

$$L[y](t) \equiv \sum_{j=0}^{n} (-1)^j \{p_j(t) y^{[j]}(t)\}^{[j]} = 0, \tag{3.1}$$

where the p_j are real-valued functions on an interval I on the real line, satisfying the hypothesis

($\mathfrak{H}_1$) $p_j \in \mathfrak{C}^j[I]$, $(j = 0, 1,..., n)$, *and* $p_n(t) > 0$ *for* $t \in I$.

As is well known, (see, for example, Reid [5, Section III: 8]), a function y is a solution of (3.1) on a subinterval I_0 of I iff y is of class $\mathfrak{C}^{2n}[I]$, and the n-dimensional vector functions $u = (u_\alpha)$, $v = (v_\alpha)$, defined by

$$\begin{aligned} &u_\alpha(t) = y^{[\alpha-1]}(t),\ \alpha = 1,..., n; \\ &v_n(t) = p_n(t) y^{[n]}(t),\ v_\beta(t) = p_\beta(t) y^{[\beta]}(t) - v'_{\beta+1}(t),\ (\beta = 1,..., n-1), \end{aligned} \tag{3.2}$$

are such that (u, v) is a solution of the system (2.18) with

$$\begin{aligned} A_{\alpha\beta}(t) &= 0 \quad \text{for} \quad \beta \neq \alpha + 1,\ A_{\alpha,\alpha+1}(t) = 1, \qquad (\alpha, \beta = 1,..., n); \\ B(t) &= \operatorname{diag}\{0, 0,..., 0, 1/p_n(t)\}; \\ C(t) &= \operatorname{diag}\{p_0(t), p_1(t),..., p_{n-1}(t)\}. \end{aligned} \tag{3.3}$$

When $y(t)$ and $(u(t), v(t))$ are related by (3.2) then

$$L[y] = p_0(t)\, u_1(t) - v_1'(t). \tag{3.4}$$

For the related system (2.18) we have $\eta \in \mathfrak{D}[a, b] : \zeta$ iff $\eta = (y^{[\alpha-1]})$, $y \in \mathfrak{A}^{n,2}[a, b]$ and $\zeta = (\zeta_\alpha(t))$ is a vector of $\mathfrak{L}_n{}^2[a, b]$ such that $\zeta_n = p_n y^{[n]}$; the added conditions requiring $\eta \in \mathfrak{D}_0[a, b] : \zeta$ are $y^{[\alpha-1]}(a) = 0 = y^{[\alpha-1]}(b)$, $(\alpha = 1,..., n)$; that is, $\eta = (y^{[\alpha-1]}) \in \mathfrak{D}_0[a, b]$ iff $y \in \mathfrak{A}_0^{n,2}[a, b]$. Moreover, if $\eta = (y^{[\alpha-1]})$ is real-valued the function $J_0[\eta; a, b]$ is equal to

$$\hat{J}_0[y; a, b] = \int_a^b \left(\sum_{j=0}^{n} p_n(t) \{y^{[j]}(t)\}^2 \right) dt \tag{3.5}$$

and if also $y \in \mathfrak{A}^{2n}[a, b]$ an integration by parts yields the relation

$$\hat{J}_0[y; a, b] = \int_a^b y(t) L[y](t)\, dt. \tag{3.6}$$

Relative to the differential system (3.1) two distinct values t_1 and t_2 are said to be *conjugate* if these values are conjugate with respect to the associated system (2.16) as defined in the preceding section. The associated system (2.16) is readily seen to be normal on arbitrary subintervals of I, and thus distinct values t_1 and t_2 are conjugate with respect to (3.1) whenever there exists a nonidentically vanishing real-valued solution y of (3.1) satisfying $y^{[\alpha-1]}(a) = 0 = y^{[\alpha-1]}(b)$, $(\alpha = 1,..., n)$.

If hypothesis $(\mathfrak{H}_1)$ is satisfied, and (3.1) is disconjugate on a compact subinterval $[a, b]$ of I, then in terms of the elements $G_{rr}(s, s)$ as defined in the preceding section for the corresponding incompatible system (2.21), one has from Theorem 2.3 a characterization of the largest values $\mu_{1r}(s)$, $a < s < b$ and μ_{1r} such that for real-valued $y \in \mathfrak{A}_0^{n,2}[a, b]$ we have

$$\hat{J}_0[y; a, b] \geqslant \mu_{1r}(s)\{y^{[r-1]}(s)\}^2, \tag{3.7}$$

$$\hat{J}_0[y; a, b] \geqslant \mu_{1r}\{y^{[r-1]}(s)\}^2, \qquad s \in (a, b). \tag{3.8}$$

In particular, $G_{11}(t, s)$ is the Green's function $g(t, s)$ for the incompatible differential system

$$L[y] = 0, \qquad y^{[\alpha-1]}(a) = 0 = y^{[\alpha-1]}(b), \quad (\alpha = 1,..., n). \tag{3.9}$$

As is well-known, (see, for example, [5, Section III: 8]), g is characterized by the following properties:

(a) *on* $[a, b] \times [a, b]$, *g is of class* $\mathfrak{C}^{[2n-2]}$, *and* $g(t, s) \equiv g(s, t)$; (3.10)

(b) *for* $s \in (a, b)$, *g as a function of t is a solution of* (3.1) *on each of the intervals* $[a, s)$, $(s, b]$, *and*

$$g^{[\alpha-1,0]}(a, s) = 0 = g^{[\alpha-1,0]}(b, s), \ (\alpha = 1,..., n),$$
$$(-1)^{n-1} p_n(s)\{g^{[2n-1,0]}(s^-, s) - g^{[2n-1,0]}(s^+, s)\} = 1.$$

These results are formalized in the following theorem.

THEOREM 3.1. *Suppose that hypothesis* $(\mathfrak{H}_1)$ *is satisfied, and* (3.1) *is disconjugate on a compact subinterval* $[a, b]$ *of* I. *Then for* G, G_0 *the* $n \times n$ *matrix functions for the incompatible vector system* (2.21) *as specified in the preceding section, we have that* $\mu_{1r}(s) = 1/G_{rr}(s, s)$ *for* $s \in (a, b)$, *and*

$$\mu_{1r} = 1/\mathrm{Max}\{G_{rr}(s, s) \mid s \in [a, b]\}$$

are the largest values for which inequalities (3.7) *and* (3.8) *hold*; *in particular,* $G_{11}(s, s) = g(s, s)$, *where* $g(t, s)$ *is the Green's function for the incompatible differential system* (3.9), *and equality holds in* (3.7) *if and only if* $y(t)$ *is of the form* $\kappa g(t, s)$ *for* $t \in [a, b]$.

As an application of the above theorem we have the following inequality of Liapunov type.

THEOREM 3.2. *Suppose that hypothesis* ($\mathfrak{H}_1$) *is satisfied, and* (3.1) *is disconjugate on a compact subinterval* $[a, b]$ *of* I. *If* $q(t)$ *is a real-valued continuous function on* $[a, b]$ *such that relative to the differential equation*

$$L[y](t) - q(t)\,y(t) = 0, \tag{3.11}$$

there exists on $[a, b]$ *a pair of conjugate points* c, d, *then* $q^+(t) = \frac{1}{2}[q(t) + |\,q(t)|]$ *must satisfy the integral condition*

$$\int_a^b q^+(t)\,dt > 1/\mathrm{Max}\{g(s, s) \mid s \in [a, b]\}, \tag{3.12}$$

where $g(t, s)$ *is the Green's function for the incompatible system* (3.9).

Suppose that c and d are conjugate relative to (3.11), where $a \leqslant c < d \leqslant b$, and let y be a nonidentically vanishing real-valued solution of (3.11) such that $y^{[\alpha-1]}(c) = 0 = y^{[\alpha-1]}(d)$, $(\alpha = 1,\ldots, n)$. If $s \in (c, d)$ is such that $y^2(t)$ assumes its maximum on $[c, d]$ at $t = s$, then

$$y^2(s) \int_a^b q^+(t)\,dt \geqslant \int_c^d q(t)\,y^2(t)\,dt = \int_c^d yL[y]\,dt = \hat{J}_0[y; c, d]. \tag{3.13}$$

Now if $w(t) = y(t)$ for $t \in [c, d]$, and $w(t) = 0$ for $t \in [a, c] \cup [d, b]$, then $w \in \mathfrak{A}_0^{n,2}[a, b]$, w is of class $\mathfrak{C}^{2n}$ in a neighborhood of $t = s$, and consequently we have

$$\hat{J}_0[y; c, d] = \hat{J}_0[w; a, b] > [1/g(s, s)]\,y^2(s). \tag{3.14}$$

Inequality (3.12) is a direct consequence of (3.13), (3.14).

4. THE SPECIAL EQUATION $(-1)^n y^{[2n]} = 0$

In case $p_n(t) \equiv 1$ and $p_j(t) \equiv 0$ for $j = 0, 1,\ldots, n - 1$, equation (3.1) becomes

$$L[y](t) \equiv (-1)^n\,y^{[2n]}(t) = 0, \tag{4.1}$$

and the u_α, v_α defined by (3.2) are

$$u_\alpha(t) = y^{[\alpha-1]}(t), \qquad v_\alpha(t) = (-1)^{n-\alpha}\,y^{[2n-\alpha]}(t), \qquad (\alpha = 1,\ldots, n). \tag{4.2}$$

Moreover, for given τ the matrix functions $U_\tau(t) = [U_{\tau,\alpha\beta}(t)]$, $V_\tau(t) = [V_{\tau,\alpha\beta}(t)]$ defined by

$$\begin{aligned}
U_{\tau,\alpha\beta}(t) &= (t-\tau)^{2n+1-\alpha-\beta}/(2n+1-\alpha-\beta)!, && (\alpha, \beta = 1,\ldots, n);\\
V_{\tau,\alpha\beta}(t) &= (-1)^{n-\alpha}\,(t-\tau)^{\alpha-\beta}/(\alpha-\beta)!, && 1 \leqslant \beta \leqslant \alpha \leqslant n,\\
&= 0, && 1 \leqslant \alpha < \beta \leqslant n,
\end{aligned} \tag{4.3}$$

are such that $(U, V) = (U_\tau, V_\tau)$ is a solution of the matrix differential equations $L_1[U, V] = 0$, $L_2[U, V] = 0$ associated with (4.1) as presented in the preceding sections. In particular, $U_\tau(\tau) = 0$, and $V_\tau(\tau)$ is nonsingular. Indeed, for $n = 1$ we have $U_\tau(t) = t - \tau$ and $V_\tau(t) = 1$. Therefore, the $n \times n$ matrix $\Delta = V_\tau(\tau)$ is equal to 1 for $n = 1$, and the diagonal matrix $\operatorname{diag}\{(-1)^{n-1},..., 1\}$ for $n > 1$. In the notation of the discussion in Section 2 following Theorem 2.2, we have that $M = V_b{}^*U_a - U_b{}^*V_a$ is equal to $\Delta U_a(b)$, and, in particular,

$$G(s, s) = -U_a(s)\, U_a^{-1}(b)\, \Delta U_b{}^*(s), \qquad \text{for} \quad a \leqslant s \leqslant b. \tag{4.4}$$

For the special case of $n = 1$, we have

$$G(s, s) = (s - a)(b - s)/(b - a),$$

and consequently the greatest value of $G(s, s)$ on $[a, b]$ is

$$G([a + b]/2, [a + b]/2) = (b - a)/4.$$

In this instance inequalities (3.8) and (3.12) reduce respectively to (1.1) and (1.3).

For the case $n = 2$, one may verify that for $s \in [a, b]$ we have

$$G_{11}(s, s) = (s - a)^3\, (b - s)^3/[3(b - a)],$$

$$G_{22}(s, s) = \frac{12(s - a)(b - s)}{(b - a)^3}\left(\frac{(s - a)^2}{4} + \frac{b - a}{12}\,[b + 2a - 3s]\right).$$

Again, the greatest values of each of these functions occurs at $s = [a + b]/2$, and

$$G_{11}([a + b]/2, [a + b]/2) = (b - a)^3/192,$$

$$G_{22}([a + b]/2, [a + b]/2) = (b - a)/16.$$

Thus for $n = 2$ and $r = 1, 2$ inequality (3.8) is real for $y \in \mathfrak{A}_0^{2,2}[a, b]$ and $s \in (a, b)$ the inequalities

$$\begin{aligned}
&\text{(a)} \qquad \int_a^b [y''(t)]^2\, dt \geqslant (192/(b - a)^3)\, y^2(s), \\
&\text{(b)} \qquad \int_a^b [y''(t)]^2\, dt \geqslant (16/(b - a))[y'(s)]^2.
\end{aligned} \tag{4.5}$$

Inequality (4.5a) was communicated to the author by C. D. Ahlbrandt, who established the result without recognizing that in his proof he was essentially computing a Green's function. Subsequently, Ahlbrandt also brought to my attention the paper of A. Ju. Levin [1], whose Theorem 4 states in the terminology of this paper that if $q(t)$ is a real-valued continuous function on $[a, b]$ such that relative to the differential equation

$$(-1)^n y^{[2n]}(t) - q(t)\, y(t) = 0,$$

there exists on $[a, b]$ a pair of conjugate points then

$$\int_a^b q^+(t)\, dt > 4^{2n-1}(2n-1)[(n-1)!]^2/(b-a)^{2n-1}. \tag{4.6}$$

As the cited paper of Levin contains no proofs, the author does not know the method by which the result was obtained. However, for $n = 1$ and $n = 2$ the respective values $4/(b-a)$ and $192/(b-a)^3$ agree with the constants appearing in the right-hand members of (1.1) and (4.5a), and thus are the reciprocals of the respective $G_{\alpha\alpha}([a+b]/2, [a+b]/2)$. Although the author has not evaluated the n-th order determinant appearing for the element $G_{11}(s, s)$ in (4.4), it is conjectured that the constant of Levin is indeed the best value for the respective inequality; that is, it is conjectured that the maximum of $G_{11}(s, s)$ on $[a, b]$ occurs at $s = [a+b]/2$, and that the value of this maximum is $(b-a)^{2n-1}/\{4^{2n-1}(2n-1)[(n-1)!]^2\}$.

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